Continuous Systems

So far we have mostly considered discrete systems $x_{n+1} = f(x_n)$ where x could be one-dimensional or multi-dimensional and part of different number systems, usually $\mathbb{R}$ or $\mathbb{C}$. Dependence can be one step back or several steps back like $x_{n+1} = f(x_n, x_{n-1}, \ldots)$. This is a **recurrence relation** that fixes a sequence once initial values of $x_0, x_1, \ldots$ are given, the same number of initials as the maximal number of steps back in the defining relation.

Discrete versus Continuous

Now we will extend to continuous systems where n goes from being an integer to a continuous parameter like t and the unknown goes from being a number to a function like $x(t)$. The equation will be a differential equation containing derivatives of the unknown function. Dependence two steps back will translate to a differential equation with derivatives of second order and so on.

Linear discrete systems like $x_t = x_{t-1} + x_{t-2}$ can be solved with closed-form expresions based on a finite number of standard mathematical operators $+, -, \times, \div, \sqrt[n]{}, \exp, \log$ and trigonometric functions. The methods used are similar to the ones used to solve linear ordinary differential equations like $x'' - x' - x = 0$.

Solve the characteristic function $r^2 - r - 1 = 0 \to \begin{cases} r_1 = 1/2 + \sqrt{5}/2 \\ r_2 = 1/2 - \sqrt{5}/2 \end{cases}$

The discrete case is solved by $x_t = C_1 r_1^t + C_2 r_2^t$
The continuous case is solved by $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
The constnats C_1, C_2 are fixed by the initial conditions x_0, x_1 or for the continuous case $x(0), x(1)$.

In the discrete case $x_0 = 0$ and $x_1 = 1$ gives the Fibonaccis sequence: 0,1,1,2,3,5,8,13,21, ... that has the closed-form solution:

$$\begin{cases} x_0 = 0 \\ x_1 = 1 \end{cases} \to \begin{cases} C_1 + C_2 = 0 \\ C_1 r_1 + C_2 r_2 = 1 \end{cases} \to \begin{cases} C_1 = 1/\sqrt{5} \\ C_2 = -1/\sqrt{5} \end{cases} \to x_t = \frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^t - \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^t$$

Linear versus Nonlinear

A **differential equation** of an unknown function $\boldsymbol{y}(x_1, x_2, \ldots, x_n)$ can be expressed with a differential operator $\mathcal{L}$ acting on $\boldsymbol{y}$ and its derivatives. If $n > 1$ these derivatives are partial derivatives like $\frac{\partial y}{\partial x_1}$ and $\frac{\partial^3 \boldsymbol{y}}{\partial^2 x_1 \partial x_2}$.
A differential equation $\mathcal{L}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x})$ is non-linear if $\mathcal{L}$ is non-linear in its effect on $\boldsymbol{y}$ with derivatives $\mathcal{L}(a_1\boldsymbol{y}_1 + a_2\boldsymbol{y}_2) \neq a_1\mathcal{L}(\boldsymbol{y}_1) + a_2\mathcal{L}(\boldsymbol{y}_2)$.

For an unknown function in one dimension $y: \mathbb{R} \to \mathbb{R}$, to be linear is to be described by a linear polynomial: $a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^n = b(x)$. When $a_k(x) \equiv a_k$ there are solutions that can be expressed with integrals. This is also the case when $n = 1$ and $a_1(x)/a_0(x)$ is non-constant.

Note that:
$x^2 y' = f(x)$ is linear since $\mathcal{L}(y_1 + y_2) = x^2 D(y_1 + y_2) = x^2 y_1{}' + x^2 y_2{}' = \mathcal{L}(y_1) + \mathcal{L}(y_2)$ but
$xy'^2 = f(x)$ is non-linear since $\mathcal{L}(y_1 + y_2) = x\big(D(y_1 + y_2)\big)^2 = x(y_1' + y_2{}')^2 = \mathcal{L}(y_1) + \mathcal{L}(y_2) + 2xy_1{}'y_2{}'$

Non-linear differential equations can only be solved with general methods in very special cases. There are seldom simple formulas bases on elementary function and integrals for their solutions. The solutions can have complicated behavior typical of chaos over extended domains of their definition.

Ordinary versus Partial

There are linear and non-linear differential equations and then there are ordinary and partial differential equations. The unknown function in an **Ordinary Differential Equation** (ODE) has only one independent variable so all the derivatives are of type d/dx and not partial deivatives $\partial/\partial x$. When the independen variable belongs to $\mathbb{R}$ it's often natural to see the function as describing some form of variation with time. Time derivatives in physics of a function $x(t)$ are sometimes denoted $\dot{x}(t)$ instead of $x'(t)$, second derivatives $\ddot{x}(t)$ and so on.

In a **Partial Differential Equation** (PDE) the unknown function has several indepenent varibles and derivatives becomes partial. Partial derivaties can be written with indices $\frac{\partial^2 u(x,y)}{\partial x \partial y} = u_{xy}$. If second order derivatives are continuous the order of derivation is irrelevant $u_{xy} = u_{yx}$.

Differential equations	Linear	Nonlinear
Ordinary	$y'' + 2y' + 3y = 4$	$\frac{dx}{dt} = r \cdot x(1-x)$
Partial	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	$\varphi_{tt} - \varphi_{xx} = \varphi - \varphi^3$

The independent variables in physicsare often space variables (x, y, z) and/or time variable (t).

A good illustration of a non-linear PDE that models nature in both its regular and chaotic aspects is the **Navier-Stokes equation**:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right) = \eta\nabla^2\boldsymbol{v} - \nabla\boldsymbol{p} + \Delta\rho g\boldsymbol{y}$$

Flow velocity, a vector at every point and time of the fluid. Zero for steady-state

Inertial term Convection

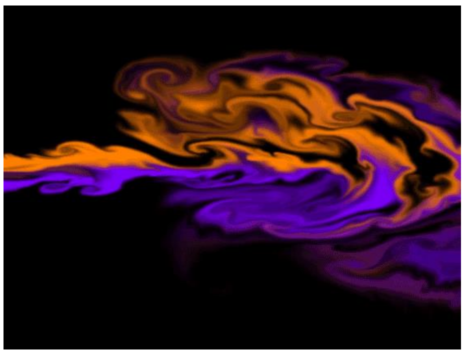
Diffusion-like viscosity term

Pressure gradient

External force on fluid

Directional vector of gravity

The equation describes the motion of viscous fluids. It can be derived from basic assumptions of conservation of momentum and mass in every part of a Newtoninan fluid, tensors for viscous stress and strain are related by a constant rather than the most general case of a tensor and the constant is independent of stress and velocity of flow. If the fluid is isotropic its viscosity reduces to two coefficients, the 1st and 2nd coefficient of viscosity.



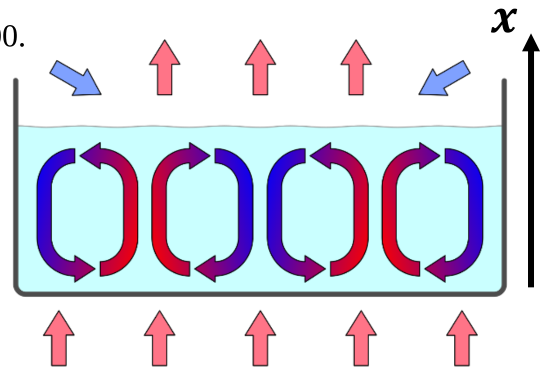
The equation describes all kinds of flows, from weather and ocean currents, to flows in pipes and air flow around a wing.

Thew phenomena of turbulence is handled by the nonlinear partial differential equation of Navier-Stokes.

From Navier-Stokes to the Lorenz equations

The Navier-Stokes PDE describes the the chaotic phenomena of turbulence in the flow of gases and liquids, but the equations are far to compl,icated to get a grip on. In 1963 Edward Lorenz, a meterologist made some simplifying assumptions for the Navier-Stokes equations in order to study atmospheric convection and in particular convection rolls in a situation with a gas bounded from top and from below by walls at different temparatures.

The situation had been studied experimentally by Henri Bénard in 1900. He used a fluid layer heated from below in a gravitational field. Fluid from below tends to rise and and cold liquid at the top 'wants ' to fall but the motion is opposed by viscous forces. When the temperature difference ΔT beween top and bottom is small viscosity wins and the liquid remains at rest while heat is transported by heat conduction. As ΔT is increased a critical stage is reached when the state becomes unstable and a convection rolls develop, they are called Bénard cells.



As ΔT is further increased the complexity and the number of convection rolls increase until a critical moment is reached and chaos sets in. Thje irregular and chaotic motion is studied by looking at the Fourier-transform of the fluid motion in the x-direction. This way multiply periodic beavior can be distinguisged from chaos

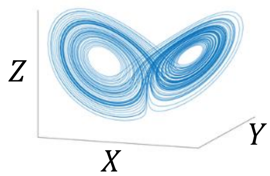
$$x(\omega) = \lim_{T\to\infty} \int_0^T e^{i\omega t} x(t) dt \quad \text{with power spectrum} \quad P(\omega) \equiv |x(\omega)|^2$$

For multiply periodic motion $P(\omega)$ consists of discrete lines at specific frequencies and when the transition to chaos occurs, $P(\omega)$ begin to show the characteristics of low frequency noise, not going down to zero between maxima. The driving parameter in the transformation from regular motion to chaos is the Rayleigh number, R.

This situation is common in many natural systems. They are modelled by a differential equation containg an external parameter controlling the trransition from regular to irregular motion, and sometimes between intervals of irregular and chaotic behavior. At the critical value of the extrenal parameter there is a phase change.

When the Navier-Stokes equation is simplified to describe the Bénard experiment it reults in a system of ordinary differential equations where the parameters X, Y, Z describe the system, they are not space variables:

$$\begin{cases} \dot{X} = \sigma(-X+Y) \\ \dot{Y} = rX - Y - XZ \\ \dot{Z} = XY - bZ \end{cases} \longrightarrow$$



Strange attractor of the Lorenz equation

The equation is only valid around the transition from heat conduction to convection rolls. The chaos described by the Lorenz equations is different from the experimental chaos found in the power spectrum of the bénard experiment when convection rolls give way to turbulence and chatic motion in the airflow.

With $r = 28$, $\sigma = 10$ and $b = 8/3$, the parameters that Lorenz used, the system has chaotic behavior with sensitive dependence on initial conditions and the orbit of (X, Y, Z) approaches an attractor with $\mathrm{Dim_H} = 2.06$. Modern chaos theory started in the 1960s when Lorenz wrote down initial values of X, Y, Z with fewer decimals than was stored in the computer. When he repeated the numerical calulation he was surprised to find that after some time the result diverged from the original. He had found sensitive dependence on initial conditions, SIC.

This was the the butterfly-effect, "the flaps of a butterfly can cause a hurricane on another continent".

Derivation of Lorenz equations from Navier-Stokes equations

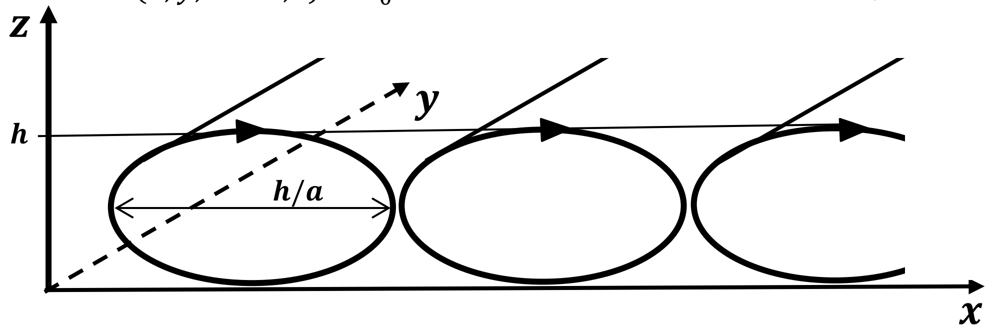
Starting from Navier-Stokes equation: $\rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}+\frac{\partial \boldsymbol{v}}{\partial t}=\boldsymbol{F}-\boldsymbol{\nabla}p+\mu\boldsymbol{\nabla}^2\boldsymbol{v}$

Heat conduction: $\frac{dT}{dt}=\kappa\boldsymbol{\nabla}^2 T$

Continuity equation: $\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})=0$

with boundary condition: $T(x,y,z=0,t)=T_0+\Delta T$, $T(x,y,z=h,t)=T_0$

$\boldsymbol{v}(\boldsymbol{r},t)$	Velocity field
$T(\boldsymbol{r},t)$	Temperature field
ρ	Density of the fluid
μ	Viscosity
$\boldsymbol{p}$	Pressure
κ	Thermal conductivity
$\boldsymbol{F}=\rho g\boldsymbol{e}_z$	External gravitational force



And some simplifying assumptions to deal with the experimental setup of the Bénard experiment:

- Translational invariance in the y-direction
- Temperature dependence of μ, p, κ can be neglected
- $\rho=\overline{\rho}(1-\alpha\Delta T)$

Gives for the continuity equation $\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0$

Introduce $\psi(x,y,z,t)$ with $v_x=-\frac{\partial\psi}{\partial z}$ and $v_z=\frac{\partial\psi}{\partial x}$ so that the continuity equation is automatically satisfied, and deviation $\theta(x,z,t)$ from linear temperature dependence $T(x,z,t)=T_0+\frac{\Delta T}{h}z+\theta(x,z,t)$.

The Navier-Stokes equation in terms of ψ and θ:

$$\begin{cases}\frac{\partial}{\partial t}\boldsymbol{\nabla}^2\psi=-\left|\frac{\partial(\psi,\boldsymbol{\nabla}^2\psi)}{\partial(x,z)}\right|+\nu\boldsymbol{\nabla}^4\psi+g\alpha\frac{\partial\theta}{\partial x}\\ \frac{\partial\theta}{\partial t}=-\left|\frac{\partial(\psi,\theta)}{\partial(x,z)}\right|+\frac{\Delta T}{h}\frac{\partial\psi}{\partial x}+\kappa\boldsymbol{\nabla}^2\theta\end{cases}$$

Where

$\left|\frac{\partial(a,b)}{\partial(x,z)}\right|\equiv\frac{\partial a}{\partial x}\cdot\frac{\partial b}{\partial z}-\frac{\partial a}{\partial z}\cdot\frac{\partial b}{\partial x}$ is the Jacobian determinant

$\boldsymbol{\nabla}^4\equiv\frac{\partial}{\partial x^4}+\frac{\partial^4}{\partial z^4}$

$\nu\equiv\frac{\mu}{\rho}$ is kinematic viscosity

Use free boundary conditions:
$\theta(0,0,t)=\theta(0,h,t)=\psi(0,0,t)=\psi(0,h,t)=\boldsymbol{\nabla}^2\psi(0,0,t)=\boldsymbol{\nabla}^2\psi(0,h,t)=0$
and keep only the lowest order terms in the Fourier expansions of ψ and θ and use the following ansatz:

$$\begin{cases}\frac{a}{1+a^2}\frac{1}{\kappa}\,\psi=\sqrt{2}\,X(t)\sin\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right)\\ \frac{\pi R}{R_c\Delta T}\theta=\sqrt{2}\,Y(t)\cos\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right)-Z(t)\sin\left(\frac{2\pi z}{h}\right)\end{cases}$$

$R\equiv\frac{g\alpha h^3}{\varkappa\nu}$ is the Rayleigh number

a is the aspect ratio (see figure above)

$R_c\equiv\frac{\pi^4(1+a^2)^3}{a^2}$

Lorenz equations

$$\begin{cases}\dot{X}=-\sigma X+\sigma Y\\ \dot{Y}=-XZ+rX-Y\\ \dot{Z}=XY-bZ\end{cases}$$

Finally in terms of the variables (X,Y,Z) which are not to be confused with (x,y,z)

Where the dot is derivative with respect to normalized time $\tau\equiv\frac{\pi^2(1+a^2)}{h^2}\kappa t$

$\sigma\equiv\nu/\kappa$ is the Prandl number, $b\equiv 4(1+a^2)^{-1}$ and $r=R/R_c\propto\Delta T$ is the external control parameter.

Ordinary differential equations

From now on focus will be on ordinary differential equations and the independent variable will represent time. The solution to the equation is a flow, time evolution of some variable or vector that represents the state of a system. The mathematical form of and ordinary differential equation is:

ODE: $\boldsymbol{F}(x, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \dots, \boldsymbol{y}^n) = \boldsymbol{0}$ or $\boldsymbol{F}\left(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}, \dots, \boldsymbol{x}^{(n)}\right) = \boldsymbol{0}$

If y is a vector then F is a vecdtor of the same dimension

In classical mechanics $\boldsymbol{x}$ will be position $\boldsymbol{r} = (x, y, z)$ and the order of the differential equation is two as in Newton's second law: $\boldsymbol{F}_{force}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = m\ddot{\boldsymbol{r}}$ or $\boldsymbol{F}(t, \boldsymbol{r}, \dot{\boldsymbol{r}}, \ddot{\boldsymbol{r}}) = \boldsymbol{0}$. Another formulation of classical is in terms of **phase space** where both position and momentum are part of the **state variable**. The dynamics is then described by an equation of first order $\boldsymbol{F}(t, \boldsymbol{u}, \dot{\boldsymbol{u}}) = 0$ with $\boldsymbol{u} = (\boldsymbol{r}, \boldsymbol{p})$, a 2·3-dimensional vector with initial state given by $\boldsymbol{r}(0)$ and $\boldsymbol{p}(0)$.

Dynamical systems

An ODE of first order can be written $d\boldsymbol{x}/dt = \boldsymbol{F}\big(\boldsymbol{x}(t)\big)$ with $\boldsymbol{x}$ an n-dimensional vector in pahase space. It's a **dynamical system** and for any initial state $\boldsymbol{x}(0)$ there is a unique future state $\boldsymbol{x}(t)$ for $t > 0$. The path followed in phase space is callled **orbit** or **trajectory**. A continuous dynamical system evolving in time is called a **flow**. It can be represented graphically by considering in one image all trajectories generated by all initial conditions.

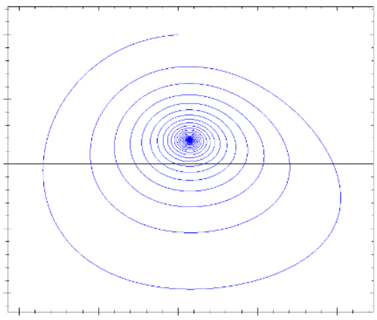
An example of a dynamical system is a forced damped pendulum:

$$\ddot{\theta} + \nu\dot{\theta} + \sin\theta = T\sin\omega t$$

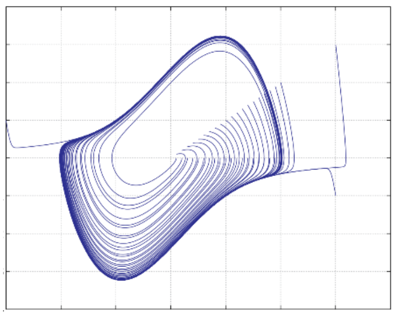
Inertia Friction Gravity Oscillating driving torque

$$\begin{cases} x_1 = \dot{\theta} \\ x_2 = \theta \\ x_3 = \omega t \end{cases} \rightarrow \begin{cases} \dot{x}_1 = T\sin x_3 - \sin x_2 - \nu x_1 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = \omega \end{cases}$$

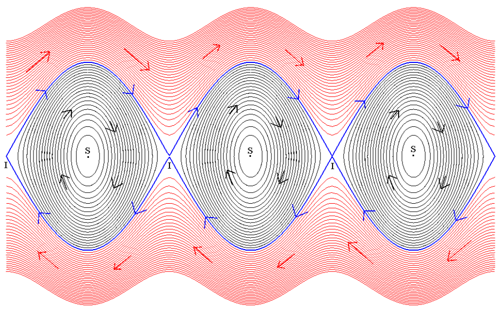
Poincaré-Bendixsons theorem on the character of limit orbits in 2-dimensional systems on a plane rules out chaos there but the pendulum system is in three dimension and the driven damped pendulum has both periodic and chaotic solutions dpending on the values of ν, T and ω.



Orbit of damped pendulum.



Flow of Van der Pol oscillator.



Flow in phase space of pendulum.

It's hard to visualize flows in dimensions higher than three. The Poincaré method can be helpful by reducing a continuous flow in N dimensions to a discrete time map by intersecting the flow with (N-1) dimensional planes. The resulting map is called the **Poincaré map**.

An example of when a Poincaré map can be helpful is the Hénon-Heiles system. It's a nonintegrable (no closed formula for its solution, it must be solved numericaly) system from classical mechanics with Hamiltonian:

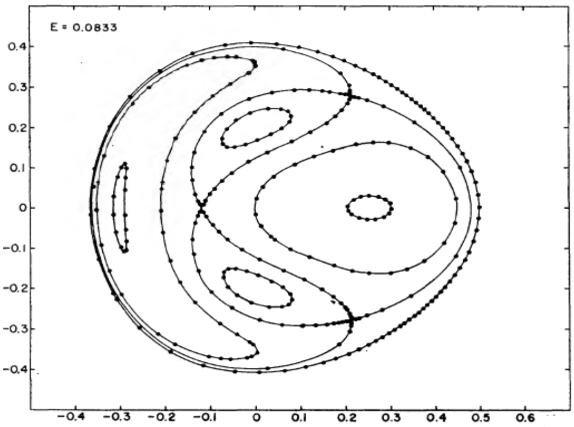
$$\mathcal{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}(x^2 + y^2) + \lambda(x^2 y - y^3/3)$$

It describes the motion of a star around a galactic center with motion restricted to a plane.

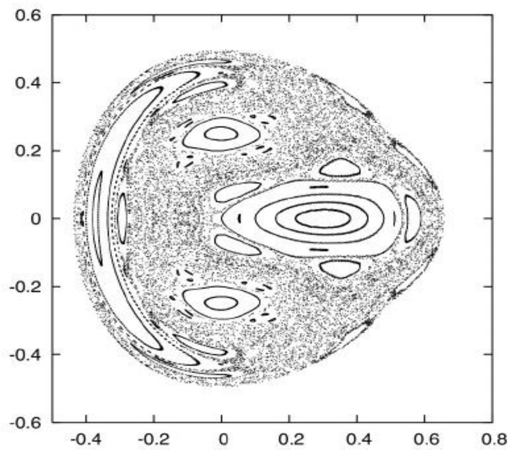
Empirical evidence suggests that a solution should have two constants of motion, total energy $E(\boldsymbol{r},t) = E_0$ and a second constant of motion $I(\boldsymbol{r},t) = I_0$. With two constants of motion an orbit should be confined to a 2-dimensional manifold.

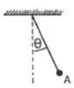
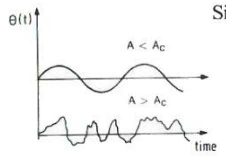
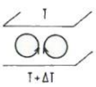
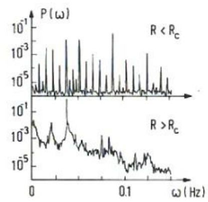
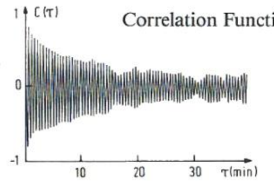
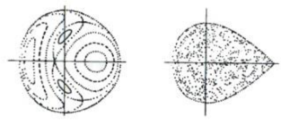
Hénon and Heiles did a Poincaré map and plotted points for which the trajectory cuts the (p_y, y)-plane. They expected that the points should form closed curves corresponding to cuts of the 2-dimensional manifold with the (p_y, y)-plane. For low energy this was the case but for high enough energy the curves became a chaotic and plane-filling scatter of points, an indication of chaotic motion in phase space and the absence of the second constant of motion.

Poincare section at $E = 0.083$



Poincare section at $E = 0.125$

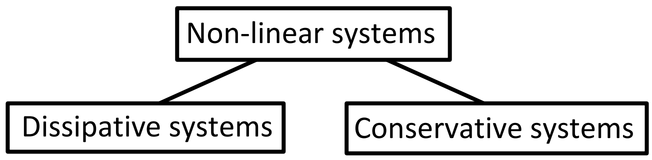


System	Equation of Motion	Indication
Pendulum 	$\ddot{\theta} + \gamma\dot{\theta} + g\sin\theta = A\cos\omega t$ $x = \theta,\ y = \dot{\theta},\ z = \omega t$ $\dot{x} = y$ $\dot{y} = -\gamma y - g\sin x + A\cos z$ $\dot{z} = \omega$	Signal 
Bénard Experiment 	$\dot{x} = -\sigma x + \sigma y$ $\dot{y} = rx - y - xz$ $\dot{z} = xy - bz$	Power Spectrum 
Belousov-Zhabotinsky Reaction $Ce_2(SO_4)_3$ ⋮ Ce^{4+}	$\dot{\vec{x}} = \vec{F}(\vec{x}, \lambda)$ $\vec{x} = [c_1, c_2, \ldots c_d]$	Correlation Function 
Hénon-Heiles System	$H = \frac{1}{2}\sum_{i=1}^{2}(p_i^2 + q_i^2) + q_1^2 q_2 - \frac{1}{3}q_2^3$ $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}},\ \dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}$	Poincaré Map 

Some examples of natural systems modeled by non-linear ordinary differential equations with an external control parameter that regulates whether the system displays chaotic evolution or not.

The third column shows different indicators of chaos. The third example comes from chemistry, it deals with concentrations of A, B, C in a chemical reaction $A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C$.

$$\begin{cases} \dot{c}_A = -k_1 c_A c_B + k_2 c_C - r(c_A - c_A(0)) \\ \dot{c}_B = -k_1 c_A c_B + k_2 c_C - r(c_B - c_B(0)) \\ \dot{c}_C = -k_1 c_A c_B - k_2 c_C - r \end{cases}$$



Nonlinear systems can be dissipative or conservative.
Dissipative systems are systems with friction and shrinkage in phase space. Chaos is indicated by strange attractors.
Conservative systems are described by a Hamiltonian and orbits with constant energy. The first and last examples in the table are conservative, the other two are dissipative.